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On the Picard-Fuchs Equations of $N = 2$ Supersymmetric E_6 Yang-Mills Theory

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Abstract

We obtain the Picard-Fuchs equations of $N = 2$ supersymmetric Yang-Mills theory with the exceptional gauge group E_6. Such equations are based on E_6 spectral curve.

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In the last few years, enormous advances have been made in understanding of the low-energy behaviour of $N = 2$ supersymmetric gauge theories. The progress was initiated with the paper of Seiberg and Witten [1], where the exact low-energy Wilsonian effective action of the pure $N = 2$ supersymmetric Yang-Mills theory with the gauge group $SU(2)$ was derived. Since then, their work has been generalized to the supersymmetric pure gauge theories with other gauge groups [2] and to the theories with the matter multiplets [3]. In principle, the exact solution of such theory is given by an algebraic curve. In the case of theories with classical Lie gauge groups, the algebraic curves is hyperelliptic [4] which of course should satisfy a set of consistency conditions. The hyperelliptic curve of the theories with the exceptional gauge groups are constructed in [5].

To understand the strong coupling region of the theory, one defines the Higgs fields $\vec{a}$ and their duals $\vec{a}_D$ which are related to the prepotential of the low-energy effective action, $F(\vec{a})$, by $\vec{a}_D = \frac{\partial F}{\partial \vec{a}}$. These fields which are periods of the Riemann surface defined by the given algebraic curve, are represented by the contour integrals of the Seiberg-Witten differential one-form λ,

$$\vec{a} = \int_{\vec{\alpha}} \lambda, \qquad \vec{a}_D = \int_{\vec{\beta}} \lambda, \tag{1}$$

where $\vec{\alpha}$ and $\vec{\beta}$ are homology cycles on the Riemann surface.

To obtain the periods, $\vec{a}$ and $\vec{a}_D$, one can derive the Picard-Fuchs (PF) operators, which annihilate $\vec{a}$ and $\vec{a}_D$. The PF equations have been derived in the case of theories with the pure classical gauge groups and also classical gauge groups with massless and massive multiplets [6]. In all these cases, the underlying algebraic curve of the theory is a hyperelliptic curve.

On the other hand in [7], the algebraic curve of the supersymmetric gauge theory is constructed from the spectral curve of the periodic Toda lattice. In the case of theories with the classical gauge groups, these curves are equivalent to the hyperelliptic curves. But recently in [8], the PF equations of the supersymmetric G_2 gauge group have been constructed from the spectral curve of the $(G_2^{(1)})^\vee$ Toda lattice theory. Also it has been shown that the calculation of $n-$instanton effects agrees with the microscopic results, while the hyperelliptic curve shows different behaviour compared with the microscopic results of [9].

Our motivation in this letter, is to shed light onto the strong coupling behaviour of E_6 theory and the comparison of the spectral curve of E_6 theory with the hyperelliptic curve of the same theory. As a first

step, we derive explicitly a set of PF equations of the theory with E_6 gauge group. In next step, we will give solutions of these equations, and the multi-instanton corrections to the prepotential of E_6 theory [10].

We use the spectral curve of E_6 given in [11],

$$\zeta + \frac{w}{\zeta} = -u_6 + \frac{q_1 + p_1\sqrt{p_2}}{x^3}, \tag{2}$$

which is obtained by degeneration of K_3 surface to an E_6 type singularity. The polynomials q_1,p_1 and p_2 are given by,

$$\begin{aligned}
q_1 &= 270x^{15} + 342u_1x^{13} + 162u_1^2x^{11} - 252u_2x^{10} + (26u_1^3 + 18u_3)x^9 \\
&- 162u_1u_2x^8 + (6u_1u_3 - 27u_4)x^7 - (30u_1^2u_2 - 36u_5)x^6 \\
&+ (27u_2^2 - 9u_1u_4)x^5 - (3u_2u_3 - 6u_1u_5)x^4 - 3u_1u_2^2x^3 \\
&- 3u_2u_5x - u_2^3; \\
p_1 &= 78x^{10} + 60u_1x^8 + 14u_1^2x^6 - 33u_2x^5 + 2u_3x^4 - 5u_1u_2x^3 \\
&- u_4x^2 - u_5x - u_2^2; \\
p_2 &= 12x^{10} + 12u_1x^8 + 4u_1^2x^6 - 12u_2x^5 + u_3x^4 - 4u_1u_2x^3 \\
&- 2u_4x^2 + 4u_5x + u_2^2,
\end{aligned} \tag{3}$$

where $u_1 \equiv c_2$, $u_2 \equiv c_5$, $u_3 \equiv c_6$, $u_4 \equiv c_8$, $u_5 \equiv c_9$, $u_6 \equiv c_{12}$ and c_i is ith Casimir of E_6 with weight i. By introducing the new variable $y = -\zeta - \frac{w}{\zeta}$, the Seiberg-Witten differential which is given by $\lambda = -2x\frac{d\zeta}{\zeta}$ becomes,

$$\lambda = \frac{xdy}{\sqrt{y^2 - 4w}}. \tag{4}$$

The derivatives of λ with respect to the Casimirs of the group are given by,

$$\begin{aligned}
\partial_i\lambda = \frac{\partial\lambda}{\partial u_i} &= -\frac{\partial_i y}{\sqrt{y^2 - 4w}} + d(*), \\
\partial_{ij}\lambda = \frac{\partial^2\lambda}{\partial u_i \partial u_j} &= \frac{\partial_i\partial_j y}{\sqrt{y^2 - 4w}} - \frac{y\partial_i y\partial_j y}{(y^2 - 4w)^{3/2}} + d(*).
\end{aligned} \tag{5}$$

Now, we take the following form for the PF operator,

$$\mathcal{L} = a_{ij}(u_1, \cdots, u_6)\partial_{ij} + a_i(u_1, \cdots, u_6)\partial_i + a. \tag{6}$$

After applying $\mathcal{L}$ to λ given in (4), we get,

$$\begin{aligned}\mathcal{L}\lambda &= \frac{L_1}{\zeta}+\frac{L_2}{x^2\zeta}+\frac{L_3}{x^3\zeta}+\frac{L_4}{x^2\zeta\sqrt{p_2}}+\frac{L_5}{x^3\zeta\sqrt{p_2}}+\frac{L_6}{x^3\zeta p_2\sqrt{p_2}}+\frac{L_7}{\zeta^3}+\frac{L_8}{x^3\zeta^3}\\ &+ \frac{L_9}{x^3\zeta^3\sqrt{p_2}}+\frac{L_{10}}{x^6\zeta^3}+\frac{L_{11}}{x^6\zeta^3\sqrt{p_2}}+\frac{L_{12}}{x^6\zeta^3 p_2}+\frac{L_{13}}{x^9\zeta^3}+\frac{L_{14}}{x^9\zeta^3\sqrt{p_2}}+\frac{L_{15}}{x^9\zeta^3 p_2},\end{aligned} \tag{7}$$

which $L_1,\cdots,L_{15}$ are complicated polynomials of $x, u_1,\cdots,u_6$ and w. The form of eq. (7) hints to choose the following form for $\mathcal{L}\lambda$,

$$\mathcal{L}\lambda = d(\frac{f(x)}{x^2\zeta\sqrt{p_2}}+\frac{g(x)}{x^2\zeta}). \tag{8}$$

To find coefficients a_{ij}, a_i and a in (6), we write the following expansion of these coefficients according to their weights, such that the weight of $\mathcal{L}$ becomes zero,

$$\begin{aligned}a_{ij} &= \sum_{p=1}^{N_{ij}}\left(d_{ij,p}\prod_{k=1}^{7}u_k^{n_k}\right),\\ a_i &= \sum_{p=1}^{N_i}\left(d_{i,p}\prod_{k=1}^{7}u_k^{n_k}\right),\end{aligned} \tag{9}$$

where N_{ij} and N_i are the number of different solutions of the relations $\{\sum_k n_k[k]=[i]+[j]\}$ and $\{\sum_k n_k[k]=[i]\}$ respectively, and $[k]$ is the weight of u_k. In eqs. (9), $d_{ij,p}$ and $d_{i,p}$ are zero weight constants and $u_7\equiv w$ has weight 24. Choosing the weight of x equal to one, eqs. (4) and (8) imply that λ has weight one. Hence, $f(x)$ and $g(x)$ are polynomials of x with degrees 20 and 15 respectively. By these considerations, the general expression of $f(x)$ and $g(x)$ are as follows,

$$f(x) = \sum_{i=0}^{20}\left(\sum_{j=1}^{N} f_{i,j}\prod_{k=1}^{7}u_k^{n_k}\right)x^i, \tag{10}$$

$$g(x) = \sum_{i=0}^{15}\left(\sum_{j=1}^{N} g_{i,j}\prod_{k=1}^{7}u_k^{n_k}\right)x^i, \tag{11}$$

where N is the number of different solutions of $\{\sum_k n_k[k]+i=20\}$ in (10) and $\{\sum_k n_k[k]+i=15\}$ in (11). By substitution of (9),(10) and (11) in (8) and after some simplifications, we obtain a polynomial of degree 120 in x that identically must be zero in x and u_i's. Then we obtain a set of nonlinear equations for 333 parameters, $d_{i,p}$, $d_{ij,p}$, $a, f_{i,j}$, $g_{i,j}$. Solutions of these equations give a set of different PF operators. As an example,

$$\mathcal{L}_1 = (u_1u_5+u_2u_3)\partial_1\partial_5+18u_2u_3u_5\partial_4\partial_6+6u_2^2u_3\partial_4^2-4u_1u_5\partial_2\partial_3, \tag{12}$$

which corresponds to the following $f(x)$ and $g(x)$,

$$\begin{aligned} f(x) &= -(36u_1u_5+18u_2u_3)x^9-(18u_1^2u_5+12u_1u_2u_3)x^7+(9u_2u_1u_5-3u_2^2u_3)x^4, \\ g(x) &= -(9u_1u_5+3u_2u_3)x^4. \end{aligned} \tag{13}$$

Other operators are as follows,

$$\begin{aligned} \mathcal{L}_2 &= -8u_1^2u_4\partial_3^2-2u_1u_2u_4\partial_3\partial_5+3u_2^2u_3\partial_4^2+u_2^2\partial_2^2-12u_2^2\partial_1\partial_4-6u_1u_4^2\partial_3\partial_6-18u_1^2u_2^2\partial_1\partial_6-18u_1u_2^3\partial_2\partial_6 \\ &- u_1u_4\partial_1\partial_4-18u_1u_2^2u_4\partial_4\partial_6-18u_1u_2^2\partial_6; \\ \mathcal{L}_3 &= u_6u_2\partial_4\partial_5+3u_2u_4^2\partial_5\partial_6+2u_2u_4\partial_2\partial_4+4u_1u_2u_4\partial_3\partial_5+u_2^2u_4\partial_5^2-3u_6u_2\partial_2\partial_6; \\ \mathcal{L}_4 &= 12u_1^2u_2^2\partial_3\partial_4-u_1u_2\partial_1\partial_2+9u_1u_2^3\partial_2\partial_6+36u_1^2u_2u_5\partial_3\partial_6-2u_1u_2u_4\partial_3\partial_5+9u_1u_2^3u_5\partial_5\partial_6+9u_1u_2^2\partial_6; \\ \mathcal{L}_5 &= 8u_1^4\partial_1\partial_3+u_1^3u_2\partial_1\partial_5+8u_1^3u_4\partial_3\partial_4+12u_1^3\partial_3+4u_1^2\partial_1^2-u_1^2u_3\partial_1\partial_4+2u_1^2u_5\partial_2\partial_4-u_1^2u_2u_4\partial_4\partial_5 \\ &+ 4u_3u_4\partial_3\partial_4-3u_3u_4\partial_1\partial_6-u_1^2u_2^2\partial_2\partial_5+3u_1^2u_4\partial_6; \\ \mathcal{L}_6 &= 9u_2u_3u_5\partial_4\partial_6+2u_3u_5\partial_3\partial_5+(27u_3u_5^2+9u_1^2u_4u_6)\partial_6^2+2u_1^2u_6\partial_4^2+3u_1^2u_2u_6\partial_5\partial_6+12u_1^3u_6\partial_3\partial_6, \end{aligned} \tag{14}$$

where the corresponding functions $f(x)$ and $g(x)$ are of degrees (10, 5), (7, 2), (13, 8), (16, 11) and (5, 0) respectively.

CONCLUSIONS

In this paper, by a systematic method, we obtain the PF equations of $N=2$ supersymmetric E_6 Yang-Mills theory. These equations are useful to obtain the periods and multi-instanton corrections to the prepotential of the theory which is based on E_6 spectral curve [10].

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